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**Uniform in ε Convergence of Defect-Correction Method
for Convection-Diffusion Problems**

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Abstract

A finite difference method is presented for a singularly perturbed convection-diffusion problem with a discretization error estimate of order $O(N^{-2} \ln^2 N)$, which holds uniformly in the singular perturbation parameter ε . The total number of points used in a two dimensional case is N^2 . The difference method is based on a defect-correction technique on a Shishkin mesh. Its efficiency in resolving exponential boundary layers is illustrated by numerical examples.

AMS Subject Classifications: 35B25, 65N06, 65N15.

KEY WORDS singularly perturbed convection-diffusion problems, defect-correction technique, finite differences, order of discretization error, a priori adapted mesh - Shishkin mesh

1 Introduction

We consider the convection-diffusion problem (1.1) with boundary conditions (1.2),

$$\mathcal{L}u = -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu = f, \quad (x, y) \in \Omega \subset \mathbb{R}^2, \quad (1.1)$$

$$u = g, \quad (x, y) \in \Gamma, \quad (1.2)$$

whose solution is driven by the velocity vector \mathbf{b} , satisfying the condition

$$\mathbf{b} = (b_1, b_2) \geq (\beta_1, \beta_2) > (0, 0). \quad (1.3)$$

The singular perturbation parameter ε , $0 < \varepsilon \leq 1$ is used to measure the relative amount of diffusion to convection. The velocity \mathbf{b} and the righthand sides f and g are sufficiently smooth functions in a bounded convex domain of polygonal type Ω . Its boundary Γ consists of inflow (Γ_-), outflow (Γ_+) and characteristic (Γ_0) parts, defined by

$$\Gamma_- = \{(x, y) \in \Gamma, \mathbf{b} \cdot \mathbf{n} < 0\},$$

$$\Gamma_+ = \{(x, y) \in \Gamma, \mathbf{b} \cdot \mathbf{n} > 0\},$$

$$\Gamma_0 = \{(x, y) \in \Gamma, \mathbf{b} \cdot \mathbf{n} = 0\}.$$

Recently, there appeared three books [16], [17], [19] dealing with the numerical solution of singularly perturbed equations, showing the interest in this type of problems.

The major difficulty in the numerical solution of (1.1)-(1.2) is to find a numerical approximation scheme which is uniformly accurate in ε and with a solution cost which does not grow with decreasing ε . The standard finite difference scheme of upwind and central type on a uniform mesh does not belong to this class. Moreover, the pointwise error is not necessarily reduced by successive uniform refinement of the mesh in contrast to solving unperturbed problems. Furthermore, the central difference scheme is numerically unstable and gives an oscillating approximate solution unless the meshsize is extremely fine. The introduction of specially adapted (a priori or a posteriori) meshes in the layer region(s) overcomes these difficulties. Bakhvalov was the first using a uniform mesh outside the layer(s) and specially graded mesh at the layer(s), see [4]. Then, Shishkin [23] introduced *piecewise equidistant meshes* and showed that one can also obtain uniform convergence in the layer in this case. A defect-correction method combined with such Shishkin mesh is the focus of our paper. It is organized as follows. The differentiability properties of the exact solution of (1.1)-(1.2) as well as its asymptotic expansion are considered in Section 2. In Section 3, the defect-correction method on a Shishkin mesh is presented. The *uniform convergence in ε* of the defect-correction solution to the exact solution u is theoretically analyzed in Section 4 and numerically verified in Section 5.

Notation 1 Throughout the paper we denote with C a generic constant, which can be different at different occurrences and is independent of ε and the mesh parameters.

Notation 2 We denote by $\|\cdot\|$ the usual L_∞ -norm.

Notation 3 Both notations u_x and $\frac{\partial u}{\partial x}$ are used to denote the partial derivative of u with respect to x .

2 Differentiability properties and asymptotic expansion of the exact solution

Consider now problem (1.1)-(1.2) in $\bar{\Omega} = [0, 1]^2$, the simplest case of polygonal domain, with homogeneous Dirichlet boundary conditions on Γ . Two exponential boundary layers occur then at the outflow boundary $\Gamma_+ = \{(x = 1, 0 < y < 1) \cup (0 < x < 1, y = 1)\}$. Condition (1.3) excludes the occurrence of internal and parabolic boundary layer(s). To avoid also layers caused by data incompatibility at the corners $T_1 = (0, 0)$, $T_2 = (1, 0)$, $T_3 = (1, 1)$, $T_4 = (0, 1)$ of the unit square the first-order compatibility conditions must be imposed,

$$f(T_l) = 0, \quad l = 1, 2, 3, 4, \quad (2.1)$$

which ensure that $u \in \mathbb{C}^{3,\alpha}(\overline{\Omega})$ if $f \in \mathbb{C}^{1,\alpha}(\overline{\Omega})$ and $\alpha \in (0, 1]$, see [9]. Here the space $\mathbb{C}^{k,\alpha}(\overline{\Omega})$ consists of all differentiable functions in $\overline{\Omega}$ whose derivatives of order k are Hölder continuous of degree α .

Definition 2.1 If $u \in \mathbb{C}^k(\overline{\Omega})$ then the semi-norm $|u|_k$ and the norm $\|u\|_k$ are defined by

$$|u|_k = \max_{i+j=k} \left(\sup_{(x,y) \in \overline{\Omega}} \left| \frac{\partial^{i+j} u}{\partial x^i \partial y^j}(x, y) \right| \right),$$

$$\|u\|_k = \max_{0 \leq s \leq k} |u|_s$$

and when $k = 0$ we simply write $\|u\|$.

Conditions (2.1) cannot ensure $u \in \mathbb{C}^{4,\alpha}(\overline{\Omega})$ required by the analysis in this paper, that is why we shall use the following lemma.

Lemma 2.1 If $\nabla b_1(T_l) = \nabla b_2(T_l) = (0, 0)$, $l = 1, \dots, 4$ and $f \in \mathbb{C}^{2,\alpha}(\overline{\Omega})$ then $u \in \mathbb{C}^{4,\alpha}(\overline{\Omega})$ if and only if the first-order compatibility conditions (2.1) and the second-order compatibility conditions

$$\varepsilon(f_{xx}(T_l) - f_{yy}(T_l)) - (b_1 f_x)(T_l) + (b_2 f_y)(T_l) = 0, \quad 1 \leq l \leq 4 \quad (2.2)$$

are satisfied. If $f \in \mathbb{C}^{3,\alpha}(\overline{\Omega})$ then $u \in \mathbb{C}^{5,\alpha}(\overline{\Omega})$ if and only if (2.1) and (2.2) hold.

Proof See Lemmas 3.1 and Theorem 3.2 in [9]. ■

The derivatives of the exact solution u are in general not bounded for $\varepsilon \rightarrow 0$. This can be shown using an asymptotic expansion of the solution and its derivatives. Various forms of such expansions have appeared in the literature, for a survey in case of parabolic layers, see [22] and in case of exponential layers, see [14]. Estimates of the higher order derivatives are also provided in these papers. The authors of [14] derived the precise compatibility conditions on f and its derivatives at the corner points for which the solution can be decomposed into a sum of smooth and layer functions. Here we present the Shishkin-Type decomposition theorem from [14], which will be used in Section 4.

Theorem 2.1 (Shishkin-Type Decomposition) Consider problem (1.1) with homogeneous Dirichlet boundary conditions on Γ . Let $f \in \mathbb{C}^{k,\alpha}(\overline{\Omega})$ for some integer $k \geq 4$ and some $\alpha \in (0, 1]$. Let $n \geq 2$ be an integer. Set

$$\mathcal{D}_i u = \frac{\partial u}{\partial y} \frac{\partial^i}{\partial x^i} \left(\frac{b_2}{b_1} \right) + u \frac{\partial^i}{\partial x^i} \left(\frac{c}{b_1} \right) \quad \text{for } i = 0, 1.$$

Suppose that f satisfies the compatibility conditions (2.1), (2.3)-(2.5).

$$\left(\frac{f}{b_1} \right)_y(T_1) = \left(\frac{f}{b_2} \right)_x(T_1), \quad \left(\left(\frac{f}{b_1} \right)_x - \mathcal{D}_0 \left(\frac{f}{b_1} \right) \right)_y(T_1) = \left(\frac{f}{b_2} \right)_{xx}(T_1), \quad (2.3)$$

$$\left(\left(\frac{f}{b_1} \right)_{xx} - \mathcal{D}_0 \left(\left(\frac{f}{b_1} \right)_x - \mathcal{D}_0 \left(\frac{f}{b_1} \right) \right) - 2\mathcal{D}_1 \left(\frac{f}{b_1} \right) \right)_y(T_1) = \left(\frac{f}{b_2} \right)_{xxx}(T_1), \quad (2.4)$$

$$\left(b_2 \left(\frac{f}{b_2} \right)_{xx} \right)_{xy}(T_1) = \left(b_1 \left(\frac{f}{b_1} \right)_{yy} \right)_{xy}(T_1) \quad \text{and if } n \geq 4 \quad b_{2,x}(T_3) = b_{1,y}(T_3). \quad (2.5)$$

Then the solution $u \in \mathbb{C}^{3,1}(\bar{\Omega})$ and has the representation

$$u = v + z_1 + z_2 + z_{12}, \quad (2.6)$$

where v is the smooth part satisfying

$$\|v\|_2 + \varepsilon \|v\|_3 \leq C, \quad (2.7)$$

while z_1 , z_2 and z_{12} represent boundary and corner layer parts satisfying the estimates

$$\left| \frac{\partial^{i+\nu} z_1(x, y)}{\partial x^i \partial y^\nu} \right| \leq C \varepsilon^{-i} e^{-\beta_1(1-x)/\varepsilon}, \quad (2.8)$$

$$\left| \frac{\partial^{\mu+j} z_2(x, y)}{\partial x^\mu \partial y^j} \right| \leq C \varepsilon^{-j} e^{-\beta_2(1-y)/\varepsilon}, \quad (2.9)$$

$$\left| \frac{\partial^{i+j} z_{12}(x, y)}{\partial x^i \partial y^j} \right| \leq C \varepsilon^{-(i+j)} e^{-\beta_1(1-x)/\varepsilon} e^{-\beta_2(1-y)/\varepsilon}, \quad (2.10)$$

for all $(x, y) \in \Omega$ and $0 \leq i, j \leq n$, $0 \leq \mu, \nu \leq k-2$. Moreover, for all $(x, y) \in \Omega$

$$|\mathcal{L}z_1(x, y)| \leq C \varepsilon e^{-\beta_1(1-x)/\varepsilon}, \quad (2.11)$$

$$|\mathcal{L}z_2(x, y)| \leq C \varepsilon e^{-\beta_2(1-y)/\varepsilon}, \quad (2.12)$$

$$|\mathcal{L}z_{12}(x, y)| \leq C \varepsilon e^{-\beta_1(1-x)/\varepsilon} e^{-\beta_2(1-y)/\varepsilon}. \quad (2.13)$$

Proof See Theorem 5.1 in [14]. ■

This theorem provides estimates for the solution of equation (1.1) and its higher order derivatives. They will be used in Subsections 4.1 and 4.2.

Remark 2.1 For the sake of completeness of our presentation we add the equations satisfied by v, z_1, z_2, z_{12} .

(1) $v = S_0 + \varepsilon S_1$, where

$$\begin{aligned} b_1 S_{0,x} + b_2 S_{0,y} + c S_0 &= f \quad \text{on } \Omega, \\ b_1 S_{1,x} + b_2 S_{1,y} + c S_1 &= \Delta S_0 \quad \text{on } \Omega, \\ S_i(0, y) &= S_i(x, 0) = 0, \quad i = 0, 1. \end{aligned}$$

(2) $z_1 = X_0 + \varepsilon X_1$, where in terms of the stretch variable $\xi = (1-x)/\varepsilon$, X_0 and X_1 satisfy

$$\begin{aligned} X_{0,\xi\xi} + b_1(1, y) X_{0,\xi} &= 0, \quad (\xi, y) \in (0, \varepsilon^{-1}) \times (0, 1), \\ X_{1,\xi\xi} + b_1(1, y) X_{1,\xi} &= b_2(1, y) X_{0,y} + c(1, y) X_0 + \xi b_{1,x}(1, y) X_{0,\xi}, \\ X_i(0, y) &= -S_i(1, y), \quad X_i(\xi, y) \longrightarrow 0 \text{ as } \xi \longrightarrow \infty \quad \text{for } i = 0, 1. \end{aligned}$$

- (3) $z_2 = Y_0 + \varepsilon Y_1$, where in terms of the stretch variable $\eta = (1 - y)/\varepsilon$, Y_0 and Y_1 satisfy

$$\begin{aligned} Y_{0,\eta\eta} + b_2(x, 1)Y_{0,\eta} &= 0, \quad (x, \eta) \in (0, 1) \times (0, \varepsilon^{-1}), \\ Y_{1,\eta\eta} + b_2(x, 1)Y_{1,\eta} &= b_1(x, 1)Y_{0,x} + c(x, 1)Y_0 + \eta b_{2,y}(x, 1)Y_{0,\eta}, \\ Y_i(x, 0) &= -S_i(x, 1), \quad Y_i(x, \eta) \longrightarrow 0 \text{ as } \eta \longrightarrow \infty \quad \text{for } i = 0, 1. \end{aligned}$$

- (4) $z_{12} = Z_0 + \varepsilon Z_1$, where in terms of the stretch variables ξ and η , Z_0 and Z_1 satisfy

$$\begin{aligned} Z_{0,\xi\xi} + Z_{0,\eta\eta} + b_1(1, 1)Z_{0,\xi} + b_2(1, 1)Z_{0,\eta} &= 0, \quad (\xi, \eta) \in (0, \varepsilon^{-1}) \times (0, \varepsilon^{-1}), \\ Z_{1,\xi\xi} + Z_{1,\eta\eta} + b_1(1, 1)Z_{1,\xi} + b_2(1, 1)Z_{1,\eta} &= \\ (\xi b_{1,x}(1, 1) + \eta b_{1,y}(1, 1))Z_{0,\xi} + (\xi b_{2,x}(1, 1) + \eta b_{2,y}(1, 1))Z_{0,\eta} + c(1, 1)Z_0, \\ Z_i(\xi, 0) &= -X_i(\xi, 1), \quad Z_i(0, \eta) = -Y_i(1, \eta), \quad Z_i(\xi, \eta) \longrightarrow 0 \text{ as } \xi, \eta \longrightarrow \infty \quad \text{for } i = 0, 1. \end{aligned}$$

Remark 2.2 Instead of estimate (2.7) we assume that the smooth part of the solution satisfies

$$\|v\|_{2+k} + \varepsilon\|v\|_{3+k} \leq C, \quad \text{for } k = 1, 2. \quad (2.14)$$

This assumption requires that (2.2) as well as extra compatibility conditions similar to (2.3)-(2.5) hold true, but we do not consider them since, in practice, the numerical problems we deal with rarely satisfy (2.2) and almost never (2.3)-(2.5). Nevertheless, the rate of convergence is satisfactory, which indicates that compatibility conditions (2.3)-(2.5) could be weakened, but this is one of the topics not solved in the literature yet.

3 Defect-correction method on a Shishkin mesh

The Shishkin mesh is defined by a tensor product of two one-dimensional piecewise equidistant meshes. Let N_x and N_y be the points in the x - and the y -direction, correspondingly. Then, we set

$$\tau_x = \min\left\{\frac{1}{2}, \frac{2}{\beta_1}\varepsilon \ln N_x\right\}, \quad \tau_y = \min\left\{\frac{1}{2}, \frac{2}{\beta_2}\varepsilon \ln N_y\right\},$$

and call $1 - \tau_x$ and $1 - \tau_y$ the transition points from the coarse to the fine mesh in the corresponding directions. The coarse and fine meshsizes are defined by

$$\begin{aligned} H_x &= (1 - \tau_x)/(N_x/2), & H_y &= (1 - \tau_y)/(N_y/2), \\ h_x &= \tau_x/(N_x/2), & h_y &= \tau_y/(N_y/2), \end{aligned}$$

and written formally

$$\Omega_x^N = \Omega_{c,x}^N \cup \Omega_{f,x}^N, \quad \text{where}$$

$$\Omega_{c,x}^N = \{x_i = iH_x, i = 0, \dots, N_x/2\}, \quad (3.1)$$

$$\Omega_{f,x}^N = \{x_i = 1 - \tau_x + (i - N_x/2)h_x, i = N_x/2 + 1, \dots, N_x\}.$$

Analogously, Ω_y^N is defined. Then, the piecewise equidistant Shishkin mesh in Ω is $\Omega^N = \Omega_x^N \times \Omega_y^N$. It is coarse on $[0, 1 - \tau_x] \times [0, 1 - \tau_y]$ and much finer near Γ_+ . For notational simplicity we assume that $N_x = N_y = N$ and $\tau_x, \tau_y < 1/2$.

Definition 3.1 The meshsizes $h_{x,i}$ and $h_{y,j}$ used in the rest of the paper are defined

$$\text{by } h_{x,i} = x_i - x_{i-1} = \begin{cases} H_x, & 1 \leq i \leq N/2, \\ h_x, & N/2 < i \leq N, \end{cases} \quad h_{y,j} = y_j - y_{j-1} = \begin{cases} H_y, & 1 \leq j \leq N/2, \\ h_y, & N/2 < j \leq N. \end{cases}$$

The *defect-correction method* has been considered in [1], [5], [10], [11], [12], among others. Two types of difference operators are used in a defect-correction technique - high order of accuracy operator but not stable and low order of accuracy operator but stable. Let $L^{(0)}$ and $L^{(1)}$, respectively, be operators of such types. They can be chosen in various ways. Given a grid function $\{v_{ij}\}$ on Ω^N , we use the following pair $(L^{(1)}, L^{(0)})$:

- (i) $L^{(1)}$ is a upwind difference operator, in which the second order derivatives of v at (x_i, y_j) are approximated by

$$D_+^x D_-^x v_{ij} = \frac{2}{h_{x,i} + h_{x,i+1}} \left(\frac{v_{i+1j} - v_{ij}}{h_{x,i+1}} - \frac{v_{ij} - v_{i-1j}}{h_{x,i}} \right) \quad (3.2)$$

$$D_+^y D_-^y v_{ij} = \frac{2}{h_{y,j} + h_{y,j+1}} \left(\frac{v_{ij+1} - v_{ij}}{h_{y,j+1}} - \frac{v_{ij} - v_{ij-1}}{h_{y,j}} \right) \quad (3.3)$$

and the first order derivatives of v at (x_i, y_j) are approximated by upwind backward differences, since $(b_1, b_2) > (0, 0)$,

$$D_-^x v_{ij} = \frac{v_{ij} - v_{i-1j}}{h_{x,i}}, \quad D_-^y v_{ij} = \frac{v_{ij} - v_{ij-1}}{h_{y,j}}.$$

- (ii) $L^{(0)}$ is a central difference operator, in which the second order derivatives of v at (x_i, y_j) are approximated by (3.2) and (3.3), respectively, and the first order derivatives are approximated by central differences

$$D_0^x v_{ij} = \frac{v_{i+1j} - v_{i-1j}}{h_{x,i+1} + h_{x,i}}, \quad D_0^y v_{ij} = \frac{v_{ij+1} - v_{ij-1}}{h_{y,j+1} + h_{y,j}}.$$

The *second order defect-correction method* includes the following steps:

- (a) Calculate an initial approximation: $L^{(1)} u_1^N = f^N$,
- (b) Solve $L^{(1)} \delta^N = f^N - L^{(0)} u_1^N$ and set $u^N = u_1^N + \delta^N$.

The righthand side of the system in (b) and its solution δ^N are referred to as a defect and correction of the method, respectively.

Notation 4 u_1^N and u^N denote the discrete solutions of problem (1.1)-(1.2) on the Shishkin mesh Ω^N , correspondingly using the upwind scheme (step (a)) and the defect-correction method.

Theoretically, the defect-correction method can be extended to a p -th order difference method (typically $p = 3, 4$) by performing a sufficient number of correction steps. Each application of a correction step increases the order of the error by a unit if the solution is a smooth function. However, the derivatives of the solution u involve powers of ε^{-1} , see Theorem 2.1, and unphysical oscillations occur in the numerical solution which gradually spread into the interior of the domain if many correction steps are performed on a uniform mesh. For further discussions on this phenomenon see [7] and [1]. Hence, when $\varepsilon \rightarrow 0$ the above order does not show up unless the local stepsize is sufficiently small in the layer region(s), a property which is only passed by a priori or a posteriori adapted meshes to the layers. The Shishkin mesh used here belongs to the class of a priori adapted meshes.

4 Convergence analysis of problems with exponential layers

The estimate for the discretization error $\|e^N\| = \|u - u^N\|$ will be based on the discrete comparison principle given in [6], [21].

Lemma 4.1 (Discrete comparison principle)

Let v^N and w^N be solutions of the problems

$$\begin{aligned} Lv^N &= f^N & \text{on } \Omega^N, \\ v^N &= g^N & \text{on } \Gamma^N, \end{aligned} \quad \begin{aligned} Lw^N &= \tilde{f}^N & \text{on } \Omega^N, \\ w^N &= \tilde{g}^N & \text{on } \Gamma^N, \end{aligned}$$

where L is a monotone operator (i.e. $Lv \geq 0$ implies $v \geq 0$), w^N , \tilde{f}^N and \tilde{g}^N are positive. If the following conditions are satisfied

$$\begin{aligned} |f^N| &\leq \tilde{f}^N, & \text{for all } (x_i, y_j) \in \Omega^N, \\ |g^N| &\leq \tilde{g}^N, & \text{for all } (x_i, y_j) \in \Gamma^N, \end{aligned}$$

then

$$|v^N| \leq w^N, \quad \text{for all } (x_i, y_j) \in \bar{\Omega}^N = \Omega^N \cup \Gamma^N.$$

The discrete function w^N is called a barrier function for v^N .

Lemma 4.2 The difference operator $L^{(1)}$ is monotone, i.e., $L^{(1)}u^N \geq 0$ implies $u^N \geq 0$.

Proof The operator $L^{(1)}$ can be written in the form:

$$L^{(1)}u_{ij}^N = \alpha_{ij}u_{ij}^N - \sum_{k,l \neq i,j} \alpha_{kl}u_{kl}^N,$$

where (x_k, y_l) are points from the stencil used to approximate $\Delta u, u_x$ and u_y at the point (x_i, y_j) . All coefficients $\{\alpha\}$ are positive, therefore the operator $L^{(1)}$ is of positive type. Furthermore, the following inequality

$$\alpha_{ij} - \sum_{k,l \neq i,j} \alpha_{kl} \geq 0,$$

holds strictly at least for one point (x_i, y_j) , which implies that the operator is of strongly positive type. Hence, it is monotone and the associated matrix is a M-matrix. ■

The estimate of the discretization error includes the following steps. First, an estimate of the truncation error, second, a construction of a suitable barrier function w^N and finally an estimate of the discretization error, based on Lemma 4.1, are done. We split u_1^N and u^N in a similar way to (2.6), i.e.,

$$u_{1,ij}^N = (\bar{v}^N + \bar{z}_1^N + \bar{z}_2^N + \bar{z}_{12}^N)_{ij}, \quad (4.1)$$

$$u_{ij}^N = (v^N + z_1^N + z_2^N + z_{12}^N)_{ij}, \quad (4.2)$$

where $\bar{v}^N, \bar{z}_1^N, \bar{z}_2^N, \bar{z}_{12}^N, v^N, z_1^N, z_2^N, z_{12}^N$ are defined on Ω^N by

$$\begin{aligned} L^{(1)} \bar{v}_{ij}^N &= (\mathcal{L}v)_{ij}, & L^{(1)} v_{ij}^N &= 2(\mathcal{L}v)_{ij} - L^{(0)} \bar{v}_{ij}^N, \\ L^{(1)} \bar{z}_{1,ij}^N &= (\mathcal{L}z_1)_{ij}, & L^{(1)} z_{1,ij}^N &= 2(\mathcal{L}z_1)_{ij} - L^{(0)} \bar{z}_{1,ij}^N, \\ L^{(1)} \bar{z}_{2,ij}^N &= (\mathcal{L}z_2)_{ij}, & L^{(1)} z_{2,ij}^N &= 2(\mathcal{L}z_2)_{ij} - L^{(0)} \bar{z}_{2,ij}^N, \\ L^{(1)} \bar{z}_{12,ij}^N &= (\mathcal{L}z_{12})_{ij}, & L^{(1)} z_{12,ij}^N &= 2(\mathcal{L}z_{12})_{ij} - L^{(0)} \bar{z}_{12,ij}^N \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} \bar{v}_{ij}^N &= v_{ij}, & \bar{z}_{1,ij}^N &= z_{1,ij}, & \bar{z}_{2,ij}^N &= z_{2,ij}, & \bar{z}_{12,ij}^N &= z_{12,ij} & \text{on } \Gamma^N, \\ v_{ij}^N &= v_{ij}, & z_{1,ij}^N &= z_{1,ij}, & z_{2,ij}^N &= z_{2,ij}, & z_{12,ij}^N &= z_{12,ij} & \text{on } \Gamma^N. \end{aligned} \quad (4.4)$$

In Subsections 4.1 and 4.2 we shall estimate the truncation errors $L^{(1)}(g - g^N)$ and the discretization errors $g - g^N$, correspondingly, where g is any of the functions v, z_1, z_2, z_{12} .

4.1 Truncation error estimates

Let g be one of the functions v, z_1, z_2, z_{12} defined in Ω and \bar{g}^N and g^N be the corresponding terms of u_1^N and u^N . Based on (4.3) we obtain for the truncation error

$$\begin{aligned} \tau_g^N &= L^{(1)}(g - g^N) = L^{(1)}(g - \bar{g}^N + \bar{g}^N - g^N) = L^{(1)}(g - \bar{g}^N) + L^{(1)}(\bar{g}^N - g^N) = \\ &L^{(1)}(g - \bar{g}^N) + L^{(0)}\bar{g}^N - \mathcal{L}g = (L^{(1)} - L^{(0)})(g - \bar{g}^N) + L^{(0)}\bar{g}^N - \mathcal{L}g + L^{(0)}(g - \bar{g}^N) \\ &= (L^{(1)} - L^{(0)})(g - \bar{g}^N) + (L^{(0)} - \mathcal{L})g, \end{aligned} \quad (4.5)$$

i.e., it is a sum of a term arising from the defect-correction method and the truncation error of the central difference approximation.

Let $L_x^{(l)}$ and $L_y^{(l)}$, $l = 0, 1$ denote the parts of the difference operator $L^{(l)}$ corresponding to the x - and the y - directions, respectively.

Lemma 4.3 Using the Taylor's expansion with integral form of the remainder and an integration by parts, we obtain the following formulas for $1 < i, j < N$:

$$\begin{aligned}
\text{(I)} \quad (D_-^x - \frac{\partial}{\partial x})g_{ij} &= -\frac{1}{h_{x,i}} \int_{x_{i-1}}^{x_i} (\xi - x_{i-1}) \frac{\partial^2 g}{\partial x^2}(\xi, y_j) d\xi \\
\text{(II)} \quad (D_0^x - \frac{\partial}{\partial x})g_{ij} &= \frac{1}{h_{x,i} + h_{x,i+1}} \left(- \int_{x_{i-1}}^{x_i} (\xi - x_{i-1}) \frac{\partial^2 g}{\partial x^2}(\xi, y_j) d\xi + \int_{x_i}^{x_{i+1}} (x_{i+1} - \xi) \frac{\partial^2 g}{\partial x^2}(\xi, y_j) d\xi \right) \stackrel{i \neq N/2}{=} \\
&\quad \frac{1}{4h_{x,i}} \left(\int_{x_{i-1}}^{x_i} (\xi - x_{i-1})^2 \frac{\partial^3 g}{\partial x^3}(\xi, y_j) d\xi + \int_{x_i}^{x_{i+1}} (x_{i+1} - \xi)^2 \frac{\partial^3 g}{\partial x^3}(\xi, y_j) d\xi \right) \\
\text{(III)} \quad (D_+^x D_-^x - \frac{\partial^2}{\partial x^2})g_{ij} &= -\frac{1}{2h_{x,i}^2} \int_{x_{i-1}}^{x_i} (\xi - x_{i-1})^2 \frac{\partial^3 g}{\partial x^3}(\xi, y_j) d\xi + \frac{1}{2h_{x,i+1}^2} \int_{x_i}^{x_{i+1}} (x_{i+1} - \xi)^2 \frac{\partial^3 g}{\partial x^3}(\xi, y_j) d\xi \stackrel{i \neq N/2}{=} \\
&\quad \frac{1}{6h_{x,i}^2} \left(\int_{x_{i-1}}^{x_i} (\xi - x_{i-1})^3 \frac{\partial^4 g}{\partial x^4}(\xi, y_j) d\xi + \int_{x_i}^{x_{i+1}} (x_{i+1} - \xi)^3 \frac{\partial^4 g}{\partial x^4}(\xi, y_j) d\xi \right)
\end{aligned}$$

Similar expressions hold for the derivatives with respect to y .

Lemma 4.4 Using Lemma 4.3 and the formulas

$$\begin{aligned}
L_x^{(0)} g_{ij} - (\mathcal{L}_x g)_{ij} &= -\varepsilon (D_+^x D_-^x - \frac{\partial^2}{\partial x^2})g_{ij} + b_{1,ij} (D_0^x - \frac{\partial}{\partial x})g_{ij} \\
L_x^{(1)} g_{ij} - (\mathcal{L}_x g)_{ij} &= -\varepsilon (D_+^x D_-^x - \frac{\partial^2}{\partial x^2})g_{ij} + b_{1,ij} (D_-^x - \frac{\partial}{\partial x})g_{ij}
\end{aligned}$$

as well as the same formulas in the y -direction, we obtain the following estimates for the truncation error of

(a) the central difference operator $(L^{(0)} - \mathcal{L})g = (L_x^{(0)} - \mathcal{L}_x)g + (L_y^{(0)} - \mathcal{L}_y)g$

$$\begin{aligned}
\bullet \quad i \neq N/2, \quad 0 < j < N \\
|L_x^{(0)} g_{ij} - (\mathcal{L}_x g)_{ij}| &\leq C \left(b_{1,ij} h_{x,i} \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^3 g}{\partial x^3}(\xi, y_j) \right| d\xi + \varepsilon h_{x,i} \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^4 g}{\partial x^4}(\xi, y_j) \right| d\xi \right)
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\bullet \quad i = N/2, \quad 0 < j < N \\
|L_x^{(0)} g_{ij} - (\mathcal{L}_x g)_{ij}| &\leq C \left(b_{1,ij} \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^2 g}{\partial x^2}(\xi, y_j) \right| d\xi + \varepsilon \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^3 g}{\partial x^3}(\xi, y_j) \right| d\xi \right)
\end{aligned} \tag{4.7}$$

- $0 < i < N, j \neq N/2$

$$|L_y^{(0)}g_{ij} - (\mathcal{L}_y g)_{ij}| \leq C \left(b_{2,ij} h_{y,j} \int_{y_{j-1}}^{y_{j+1}} \left| \frac{\partial^3 g}{\partial y^3}(x_i, \eta) \right| d\eta + \varepsilon h_{y,j} \int_{y_{j-1}}^{y_{j+1}} \left| \frac{\partial^4 g}{\partial y^4}(x_i, \eta) \right| d\eta \right) \quad (4.8)$$

- $0 < i < N, j = N/2$

$$|L_y^{(0)}g_{ij} - (\mathcal{L}_y g)_{ij}| \leq C \left(b_{2,ij} \int_{y_{j-1}}^{y_{j+1}} \left| \frac{\partial^2 g}{\partial y^2}(x_i, \eta) \right| d\eta + \varepsilon \int_{y_{j-1}}^{y_{j+1}} \left| \frac{\partial^3 g}{\partial y^3}(x_i, \eta) \right| d\eta \right) \quad (4.9)$$

(b) the upwind operator, $(L^{(1)} - \mathcal{L})g = (L_x^{(1)} - \mathcal{L}_x)g + (L_y^{(1)} - \mathcal{L}_y)g$

- $0 < i, j < N$

$$|L_x^{(1)}g_{ij} - (\mathcal{L}_x g)_{ij}| \leq C \left(b_{1,ij} \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^2 g}{\partial x^2}(\xi, y_j) \right| d\xi + \varepsilon \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^3 g}{\partial x^3}(\xi, y_j) \right| d\xi \right) \quad (4.10)$$

$$|L_y^{(1)}g_{ij} - (\mathcal{L}_y g)_{ij}| \leq C \left(b_{2,ij} \int_{y_{j-1}}^{y_{j+1}} \left| \frac{\partial^2 g}{\partial y^2}(x_i, \eta) \right| d\eta + \varepsilon \int_{y_{j-1}}^{y_{j+1}} \left| \frac{\partial^3 g}{\partial y^3}(x_i, \eta) \right| d\eta \right) \quad (4.11)$$

Lemma 4.5 There is a constant C , independent of ε , such that

$$\int_{x_{i-1}}^{x_{i+1}} \frac{1}{\varepsilon} e^{-\beta_1(1-\xi)/\varepsilon} d\xi \leq \begin{cases} C \sinh(\frac{\beta_1 h_{x,i}}{\varepsilon}) e^{-\beta_1(1-x_i)/\varepsilon}, & i \neq N/2, \\ C e^{-\beta_1(1-x_i)/\varepsilon}, & i = N/2. \end{cases}$$

Proof $\int_{x_{i-1}}^{x_{i+1}} \frac{1}{\varepsilon} e^{-\beta_1(1-\xi)/\varepsilon} d\xi = \frac{1}{\beta_1} e^{-\beta_1(1-\xi)/\varepsilon} \Big|_{x_{i-1}}^{x_{i+1}} = \frac{1}{\beta_1} \left(e^{\frac{\beta_1 h_{x,i+1}}{\varepsilon}} - e^{-\frac{\beta_1 h_{x,i}}{\varepsilon}} \right) e^{-\beta_1(1-x_i)/\varepsilon}$

i) $i \neq N/2 \implies h_{x,i+1} = h_{x,i} \implies e^{\frac{\beta_1 h_{x,i+1}}{\varepsilon}} - e^{-\frac{\beta_1 h_{x,i}}{\varepsilon}} = 2 \sinh(\frac{\beta_1 h_{x,i}}{\varepsilon}) \implies$

$$\int_{x_{i-1}}^{x_{i+1}} \frac{1}{\varepsilon} e^{-\beta_1(1-\xi)/\varepsilon} d\xi \leq C \sinh(\frac{\beta_1 h_{x,i}}{\varepsilon}) e^{-\beta_1(1-x_i)/\varepsilon}.$$

ii) $i = N/2 \implies h_{x,i+1} = h_x, h_{x,i} = H_x, e^{\frac{\beta_1 h_{x,i+1}}{\varepsilon}} - e^{-\frac{\beta_1 h_{x,i}}{\varepsilon}} = N^{\frac{4}{N}} - e^{-\frac{\beta_1 H_x}{\varepsilon}} \leq$
 $C \implies$

$$\int_{x_{i-1}}^{x_{i+1}} \frac{1}{\varepsilon} e^{-\beta_1(1-\xi)/\varepsilon} d\xi \leq C e^{-\beta_1(1-x_i)/\varepsilon}.$$

■

Lemma 4.6 *The following estimates for the central difference term $(L^{(0)} - \mathcal{L})g$, $g \in \{v, z_1, z_2, z_{12}\}$ of the truncation error (4.5) hold for all i and j such that $0 < i, j < N$, $i, j \neq N/2$*

- (a) $|(L^{(0)} - \mathcal{L})v_{ij}| \leq C(h_{x,i}^2 + h_{y,j}^2)$
- (b) $|(L^{(0)} - \mathcal{L})z_{1,ij}| \leq C(\frac{h_{x,i}}{\varepsilon^2} \sinh(\frac{\beta_1 h_{x,i}}{\varepsilon}) e^{-\beta_1(1-x_i)/\varepsilon} + h_{y,j}^2)$
- (c) $|(L^{(0)} - \mathcal{L})z_{2,ij}| \leq C(h_{x,i}^2 + \frac{h_{y,j}}{\varepsilon^2} \sinh(\frac{\beta_2 h_{y,j}}{\varepsilon}) e^{-\beta_2(1-y_j)/\varepsilon})$
- (d) $|(L^{(0)} - \mathcal{L})z_{12,ij}| \leq \frac{C}{\varepsilon^2} (h_{x,i} \sinh(\frac{\beta_1 h_{x,i}}{\varepsilon}) + h_{y,j} \sinh(\frac{\beta_2 h_{y,j}}{\varepsilon})) e^{-(\beta_1(1-x_i) + \beta_2(1-y_j))/\varepsilon}$

If $i = \frac{N}{2}$ or/and $j = \frac{N}{2}$ then the corresponding \sinh should be replaced by a constant smaller than 1 and the order of the estimate (a) reduces to 1.

Proof The estimates above are directly obtained from estimates (4.6)-(4.9) in Lemma 4.4, estimates (2.8)-(2.10) in Theorem 2.1 and Lemma 4.5. The assumption (2.14) is used in the proof of (a). \blacksquare

Lemma 4.7 *Let g be any of the functions v, z_1, z_2, z_{12} . There are constants C_1 and C_2 , independent of the meshsizes and ε , such that for all $0 < i, j < N$*

$$|L^{(1)}(L^{(1)} - L^{(0)})(g - \bar{g}^N)_{ij}| \leq C_1 h_{x,i} \left| \left((L^{(1)} - \mathcal{L}) \frac{\partial^2 g}{\partial x^2} \right)_{ij} \right| + C_2 h_{y,j} \left| \left((L^{(1)} - \mathcal{L}) \frac{\partial^2 g}{\partial y^2} \right)_{ij} \right|.$$

Proof Let us define $\bar{e}^N = g - \bar{g}^N$. Using the formulas

$$(L_x^{(1)} - L_x^{(0)})\bar{e}_{ij}^N = b_{1,ij}(D_-^x - D_0^x)\bar{e}_{ij}^N,$$

$$L^{(1)}\bar{e}_{ij}^N = L^{(1)}(g - \bar{g}^N)_{ij} = (L^{(1)} - \mathcal{L})g_{ij}$$

(in the second one (4.3) is used) and Lemma 4.3 we obtain

$$\begin{aligned} |L^{(1)}(L_x^{(1)} - L_x^{(0)})\bar{e}_{ij}^N| &= |b_{1,ij}(L^{(1)} - \mathcal{L})(D_-^x - D_0^x)g_{ij}| = \\ &= \left| -\frac{b_{1,ij}(L^{(1)} - \mathcal{L})}{h_{x,i}(h_{x,i} + h_{x,i+1})} \left(h_{x,i+1} \int_{x_{i-1}}^{x_i} (\xi - x_{i-1}) \frac{\partial^2 g}{\partial x^2}(\xi, y_j) d\xi + h_{x,i} \int_{x_i}^{x_{i+1}} (x_{i+1} - \xi) \frac{\partial^2 g}{\partial x^2}(\xi, y_j) d\xi \right) \right| \\ &\leq \frac{b_{1,ij} h_{x,i+1}}{h_{x,i} + h_{x,i+1}} \int_{x_{i-1}}^{x_{i+1}} \left| (L^{(1)} - \mathcal{L}) \frac{\partial^2 g}{\partial x^2}(\xi, y_j) \right| d\xi \leq C_1 h_{x,i} \left| \left((L^{(1)} - \mathcal{L}) \frac{\partial^2 g}{\partial x^2} \right)(x_i, y_j) \right| \end{aligned}$$

We have a similar estimate for $|L^{(1)}(L_y^{(1)} - L_y^{(0)})\bar{e}_{ij}^N|$ and this combined with the triangular inequality completes the proof. \blacksquare

Corollary 4.1 *There is a constant C , independent of the meshsizes and ε , such that*

$$\begin{aligned}
(a) \quad & |L^{(1)}(L^{(1)} - L^{(0)})(v - \bar{v}^N)_{ij}| \leq C(h_{x,i}^2 + h_{y,j}^2) \\
(b) \quad & |L^{(1)}(L^{(1)} - L^{(0)})(z_1 - \bar{z}_1^N)_{ij}| \leq \frac{C}{\varepsilon} \left(\frac{h_{x,i}}{\varepsilon^2} + h_{y,j} \right) \sinh\left(\frac{\beta_1 h_{x,i}}{\varepsilon}\right) e^{-\beta_1(1-x_i)/\varepsilon} \\
(c) \quad & |L^{(1)}(L^{(1)} - L^{(0)})(z_2 - \bar{z}_2^N)_{ij}| \leq \frac{C}{\varepsilon} \left(h_{x,i} + \frac{h_{y,j}}{\varepsilon^2} \right) \sinh\left(\frac{\beta_2 h_{y,j}}{\varepsilon}\right) e^{-\beta_2(1-y_j)/\varepsilon} \\
(d) \quad & |L^{(1)}(L^{(1)} - L^{(0)})(z_{12} - \bar{z}_{12}^N)_{ij}| \leq \frac{C}{\varepsilon^3} \left(h_{x,i} \sinh\left(\frac{\beta_1 h_{x,i}}{\varepsilon}\right) + h_{y,j} \sinh\left(\frac{\beta_2 h_{y,j}}{\varepsilon}\right) \right) e^{-(\beta_1(1-x_i) + \beta_2(1-y_j))/\varepsilon}
\end{aligned}$$

for all i and j such that $0 < i, j < N$ and for (b), (c) and (d) $i, j \neq N/2$. If $i = \frac{N}{2}$ or/and $j = \frac{N}{2}$ then the corresponding \sinh should be replaced by a constant smaller than 1.

Proof

(a) Let g be v . Then, using Lemma 4.4(b) and (2.14) we obtain

$$\left| \left((L^{(1)} - \mathcal{L}) \frac{\partial^2 v}{\partial x^2} \right) (x_i, y_j) \right| \leq C h_{x,i} (\|v\|_4 + \varepsilon \|v\|_5) \leq C h_{x,i}$$

and an analogous estimate for $(L^{(1)} - \mathcal{L}) \frac{\partial^2 v}{\partial y^2}$. Therefore, based on Lemma 4.7 we have

$$|L^{(1)}(L^{(1)} - L^{(0)})(v - \bar{v}^N)_{ij}| \leq C(h_{x,i}^2 + h_{y,j}^2).$$

(b) Let g be z_1 and $i, j \neq N/2$. Then, using Lemma 4.4(b), (2.8) and Lemma 4.5 we obtain

$$\left| \left((L^{(1)} - \mathcal{L}) \frac{\partial^2 z_1}{\partial x^2} \right) (x_i, y_j) \right| \leq \frac{C}{\varepsilon^4} \int_{x_{i-1}}^{x_{i+1}} e^{-\frac{\beta_1(1-\xi)}{\varepsilon}} d\xi = \frac{C}{\varepsilon^3} \sinh\left(\frac{\beta_1 h_{x,i}}{\varepsilon}\right) e^{-\beta_1(1-x_i)/\varepsilon},$$

$$\left| \left((L^{(1)} - \mathcal{L}) \frac{\partial^2 z_1}{\partial y^2} \right) (x_i, y_j) \right| \leq \frac{C}{\varepsilon^2} \int_{x_{i-1}}^{x_{i+1}} e^{-\frac{\beta_1(1-\xi)}{\varepsilon}} d\xi = \frac{C}{\varepsilon} \sinh\left(\frac{\beta_1 h_{x,i}}{\varepsilon}\right) e^{-\beta_1(1-x_i)/\varepsilon}.$$

Therefore, based on Lemma 4.7 we have

$$|L^{(1)}(L^{(1)} - L^{(0)})(z_1 - \bar{z}_1^N)_{ij}| \leq \frac{C}{\varepsilon} \left(\frac{h_{x,i}}{\varepsilon^2} + h_{y,j} \right) \sinh\left(\frac{\beta_1 h_{x,i}}{\varepsilon}\right) e^{-\beta_1(1-x_i)/\varepsilon}.$$

If $i = \frac{N}{2}$ or/and $j = \frac{N}{2}$ Lemma 4.5 shows that \sinh should be replaced by a constant smaller than 1.

(c) Let g be z_2 . The proof is analogous to the proof of (b).

(d) Let g be z_{12} and $i, j \neq N/2$. Then, using Lemma 4.4(b), (2.10) and Lemma 4.5 we obtain

$$\left| \left((L^{(1)} - \mathcal{L}) \frac{\partial^2 z_{12}}{\partial x^2} \right) (x_i, y_j) \right| \leq \frac{C}{\varepsilon^3} \left(\sinh\left(\frac{\beta_1 h_{x,i}}{\varepsilon}\right) + \sinh\left(\frac{\beta_2 h_{y,j}}{\varepsilon}\right) \right) e^{-(\beta_1(1-x_i) + \beta_2(1-y_j))/\varepsilon}$$

and the same estimate holds for $\left| \left((L^{(1)} - \mathcal{L}) \frac{\partial^2 z_{12}}{\partial y^2} \right) (x_i, y_j) \right|$. Using the fact that

$$h_{x,i} \sinh\left(\frac{\beta_2 h_{y,j}}{\varepsilon}\right) + h_{y,j} \sinh\left(\frac{\beta_1 h_{x,i}}{\varepsilon}\right) \leq \begin{cases} (1 + \frac{\beta_2}{\beta_1}) h_{y,j} \sinh\left(\frac{\beta_2 h_{y,j}}{\varepsilon}\right), & \beta_1 h_{x,i} \leq \beta_2 h_{y,j}, \\ (1 + \frac{\beta_1}{\beta_2}) h_{x,i} \sinh\left(\frac{\beta_1 h_{x,i}}{\varepsilon}\right), & \beta_1 h_{x,i} > \beta_2 h_{y,j}, \end{cases}$$

and Lemma 4.7, we obtain the estimate (d). ■

4.2 Discretization error estimates

We shall consider each term of the righthand side of the following inequality separately

$$\|u - u^N\| \leq \|v - v^N\| + \|z_1 - z_1^N\| + \|z_2 - z_2^N\| + \|z_{12} - z_{12}^N\|.$$

(1) Smooth part of the solution

Corollary 4.1(a) gives us

$$|L^{(1)}(L^{(1)} - L^{(0)})(v - \bar{v}^N)_{ij}| \leq CN^{-2} \quad \text{on } \Omega^N.$$

Choosing the barrier function

$$\bar{w}_{ij} = N^{-2}(1 + x_i + y_j)$$

and using that $\bar{v}^N = v$ on Γ^N by (4.4) as well as Lemma 4.1 we have

$$|(L^{(1)} - L^{(0)})(v - \bar{v}^N)_{ij}| \leq \bar{w}_{ij} \leq CN^{-2} \quad \text{on } \bar{\Omega}^N. \quad (4.12)$$

This result combined with Lemma 4.6(a) and (4.5) yields

$$|L^{(1)}(v - v^N)_{ij}| \leq \begin{cases} CN^{-2}, & i, j \neq N/2, \\ CN^{-1}, & i = N/2 \text{ or/and } j = N/2. \end{cases}$$

We choose the barrier function

$$w_{ij} = \begin{cases} N^{-2}(1 + x_i + y_j), & i, j \neq N/2, \\ h_{x,i}^2 + h_{y,j}^2, & i = N/2 \text{ or/and } j = N/2. \end{cases}$$

Lemma 4.8 *Let us assume that*

$$N(\ln N)^{-1} \geq \max\{2\|b_1\|/\beta_1, 2\|b_2\|/\beta_2\}, \quad (4.13)$$

then there are constants C_1 and C_2 , independent of the meshsizes and ε , such that for $0 < i, j < N$

$$L^{(1)}w_{ij} \geq \begin{cases} C_1 N^{-2}, & i, j \neq N/2, \\ C_2 N^{-1}, & i = N/2 \text{ or/and } j = N/2. \end{cases}$$

Proof In case $i, j \neq N/2$, simple calculations show that $L^{(1)}w_{ij} \geq C_1 N^{-2}$. The case $i = N/2$ or/and $j = N/2$ is more interesting. First, we consider $L_x^{(1)}w_{ij}$. Then, $h_{x,i+1} = h_x$, $h_{x,i-1} = h_{x,i} = H_x$ and under the assumption (4.13) $\frac{2\varepsilon}{h_x} \geq \|b_1\|$. Hence,

$$\begin{aligned} L_x^{(1)}w_{ij} &= -\frac{2\varepsilon}{h_{x,i}+h_{x,i+1}} \left(\frac{h_{x,i+1}^2 - h_{x,i}^2}{h_{x,i+1}} - \frac{h_{x,i}^2 - h_{x,i-1}^2}{h_{x,i}} \right) + b_{1,ij} \frac{h_{x,i}^2 - h_{x,i-1}^2}{h_{x,i}} + cw_{ij} \\ &= \frac{2\varepsilon}{h_x} (H_x - h_x) + cw_{ij} \geq \|b_1\| (H_x - h_x) + cw_{ij} \geq C_2 N^{-1}. \end{aligned}$$

An analogous estimate is obtained for $L_y^{(1)}w_{ij}$ and this completes the proof. \blacksquare
Based on Lemma 4.8 we obtain

$$L^{(1)}w_{ij} \geq |L^{(1)}(v - v^N)_{ij}| \quad \text{on } \Omega^N$$

and using that $v^N = v$ on Γ^N and Lemma 4.1 we have

$$|(v - v^N)_{ij}| \leq w_{ij} \leq CN^{-2} \quad \text{on } \bar{\Omega}^N. \quad (4.14)$$

(2) Boundary layer part(s)

The discrete equivalent of the boundary value term $e^{-\beta_1(1-x)/\varepsilon}$ is given by the function

$$\begin{cases} S_{ij} = \prod_{k=i+1}^N (1 + \frac{\beta_1 h_{x,k}}{\varepsilon})^{-1}, & i = 0, \dots, N-1, \quad j = 0, \dots, N, \\ S_{Nj} = 1, & j = 0, 1, \dots, N. \end{cases}$$

The next lemma will be used to convert the truncation error bounds in Subsection 4.1 into bounds for the discretization error.

Lemma 4.9 *There is a constant C , independent of the meshsizes and ε , such that the grid function S_{ij} satisfies the following inequalities for $j = 0, \dots, N$:*

$$L^{(1)}S_{ij} \geq \frac{C}{\max\{h_{x,i}, \varepsilon\}} S_{ij}, \quad \text{for } i = 1, \dots, N-1, \quad (4.15)$$

$$e^{-\frac{\beta_1(1-x_i)}{\varepsilon}} \leq S_{ij}, \quad \text{for } i = 0, \dots, N, \quad (4.16)$$

$$S_{ij} \leq CN^{-2}, \quad \text{for } i = 0, \dots, N/2. \quad (4.17)$$

Proof

- The proof of (4.15) is given in [13] and [25].
- $e^{-\frac{\beta_1(1-x_i)}{\varepsilon}} = \prod_{k=i+1}^N e^{-\frac{\beta_1 h_{x,k}}{\varepsilon}} \leq \prod_{k=i+1}^N (1 + \frac{\beta_1 h_{x,k}}{\varepsilon})^{-1} = S_{ij}, \quad \text{for } i = 0, \dots, N-1$
and $e^{-\frac{\beta_1(1-x_N)}{\varepsilon}} = 1 = S_{Nj}$, which completes the proof of (4.16).

- Let $F(z) = \frac{1}{1+z} e^{\frac{z}{1+z}}$, $z \geq 0$. The function $F(z) \in \mathbb{C}(\mathbb{R}^+)$, $F(0) = 1$, $\lim_{z \rightarrow \infty} F(z) = 0$, therefore it is bounded, i.e., $0 < F(z) \leq C$. Let $z = \frac{\beta_1 h_{x,k}}{\varepsilon}$, hence,

$$\left(1 + \frac{\beta_1 h_{x,k}}{\varepsilon}\right)^{-1} \leq C e^{-\frac{\beta_1 h_{x,k}}{\varepsilon + \beta_1 h_{x,k}}} \implies S_{ij} \leq C e^{-\frac{\beta_1(1-x_i)}{\varepsilon + \beta_1 h_x}}, \quad \text{for } i = N/2, \dots, N,$$

and in particular for $i = N/2$ we have

$$S_{\frac{N}{2}j} \leq C e^{-\frac{\beta_1(1-x_{N/2})}{\varepsilon + \beta_1 h_x}} = C e^{-\frac{\beta_1 \tau_x}{\varepsilon + \beta_1 h_x}} = C N^{-\frac{2}{1+4N-1 \ln N}} = C N^{-2} \gamma(N) \leq C N^{-2},$$

since the function $\gamma(N) = N^{\frac{8N-1 \ln N}{1+4N-1 \ln N}}$ is bounded, i.e., $\gamma(N) \leq 9.68 = \gamma(2^4)$. Moreover, S_{ij} is an increasing function and therefore

$$S_{ij} \leq S_{\frac{N}{2}j} \leq C N^{-2}, \quad \text{for } i = 0, \dots, N/2.$$

This completes the proof of (4.17). ■

Corollary 4.2 *There are constants C_1 and C_2 , independent of the meshsizes and ε , such that for all $0 \leq j \leq N$*

$$|(L^{(1)} - L^{(0)})(z_1 - \bar{z}_1^N)_{ij}| \leq \begin{cases} C_1 \frac{H_x}{\varepsilon} (\frac{H_x}{\varepsilon^2} + h_{y,j}) S_{\frac{N}{2}j}, & 0 \leq i \leq N/2, \\ C_2 (\frac{h_x}{\varepsilon^2} + h_{y,j}) \sinh(\frac{\beta_1 h_x}{\varepsilon}) S_{ij}, & N/2 < i \leq N, \end{cases}$$

Proof Let $0 < i < N/2$, then Corollary 4.1(b) and Lemma 4.9 yield

$$\begin{aligned} |L^{(1)}(L^{(1)} - L^{(0)})(z_1 - \bar{z}_1^N)_{ij}| &\leq \frac{C_1}{\varepsilon} (\frac{H_x}{\varepsilon^2} + h_{y,j}) \sinh(\frac{\beta_1 H_x}{\varepsilon}) e^{-\frac{\beta_1 H_x}{\varepsilon}} e^{-\frac{\beta_1(1-x_{i+1})}{\varepsilon}} \\ &\leq C_1 \frac{H_x}{\varepsilon} (\frac{H_x}{\varepsilon^2} + h_{y,j}) L^{(1)} S_{\frac{N}{2}j} \end{aligned}$$

For $i = N/2$ we have

$$|L^{(1)}(L^{(1)} - L^{(0)})(z_1 - \bar{z}_1^N)_{ij}| \leq \frac{C}{\varepsilon} (\frac{H_x}{\varepsilon^2} + h_{y,j}) e^{-\frac{\beta_1(1-x_{N/2})}{\varepsilon}} \leq \frac{C H_x}{\varepsilon} (\frac{H_x}{\varepsilon^2} + h_{y,j}) L^{(1)} S_{\frac{N}{2}j}$$

and for $N/2 < i < N$

$$|L^{(1)}(L^{(1)} - L^{(0)})(z_1 - \bar{z}_1^N)_{ij}| \leq C (\frac{h_x}{\varepsilon^2} + h_{y,j}) \sinh(\frac{\beta_1 h_x}{\varepsilon}) L^{(1)} S_{ij}$$

Moreover,

$$|(L^{(1)} - L^{(0)})(z_1 - \bar{z}_1^N)_{ij}| = 0 \quad \text{on } \Gamma^N$$

and the discrete comparison principle gives the result stated in this lemma. ■

Lemma 4.10 *There is a constant C , independent of the meshsizes and ε , such that*

$$|(L^{(1)} - L^{(0)})\bar{z}_{1,ij}^N| \leq C S_{ij} \quad \text{on } \Omega^N.$$

Proof Let us apply $L^{(1)}$ to $(L^{(1)} - L^{(0)})\bar{z}_{1,ij}^N$ and consider first

$$L^{(1)}(L_x^{(1)} - L_x^{(0)})\bar{z}_{1,ij}^N = b_{1,ij}L^{(1)}(D_-^x - D_0^x)\bar{z}_{1,ij}^N = b_{1,ij}(D_-^x(\mathcal{L}z_1)_{ij} - D_0^x(\mathcal{L}z_1)_{ij}).$$

We have used (4.3) above. Further, we consider two cases:

(i) $i \neq N/2, 0 < j < N \implies h_{x,i+1} = h_{x,i}$.

Then, using (2.11), Lemma 4.9 and $S_{i+1j} = \frac{\varepsilon + \beta_1 h_{x,i+1}}{\varepsilon} S_{ij}$ we have

$$\begin{aligned} |L^{(1)}(L_x^{(1)} - L_x^{(0)})\bar{z}_{1,ij}^N| &= \left| -\frac{b_{1,ij}}{2} \frac{(\mathcal{L}z_1)_{i+1j} - 2(\mathcal{L}z_1)_{ij} + (\mathcal{L}z_1)_{i-1j}}{h_{x,i}} \right| \leq \\ &\frac{C\varepsilon}{h_{x,i}} (1 + 2e^{-\frac{\beta_1 h_{x,i}}{\varepsilon}} + e^{-\frac{2\beta_1 h_{x,i}}{\varepsilon}}) e^{-\frac{\beta_1(1-x_{i+1})}{\varepsilon}} \leq \frac{C\varepsilon}{h_{x,i}} S_{i+1j} = \frac{C\varepsilon}{h_{x,i}} \frac{(\varepsilon + \beta_1 h_{x,i})}{\varepsilon} S_{ij} \leq CL^{(1)} S_{ij}. \end{aligned}$$

(ii) $i = N/2, 0 < j < N \implies h_{x,i+1} = h_x, h_{x,i} = H_x$. Analogous to (i) we have

$$\begin{aligned} |L^{(1)}(L_x^{(1)} - L_x^{(0)})\bar{z}_{1,ij}^N| &\leq C\varepsilon \left(\frac{1}{H_x} (1 + e^{-\frac{\beta_1 H_x}{\varepsilon}}) + \frac{1}{H_x + h_x} (e^{\frac{\beta_1 h_x}{\varepsilon}} + e^{-\frac{\beta_1 H_x}{\varepsilon}}) \right) e^{-\frac{\beta_1(1-x_i)}{\varepsilon}} \leq \\ &\frac{C\varepsilon}{H_x} S_{ij} \leq CL^{(1)} S_{ij}. \end{aligned}$$

Above we have used that $e^{\frac{\beta_1 h_x}{\varepsilon}} = N^{4/N} \longrightarrow 1$ as N is a big number.

Second, we consider

$$\begin{aligned} |L^{(1)}(L_y^{(1)} - L_y^{(0)})\bar{z}_{1,ij}^N| &= |b_{2,ij}L^{(1)}(D_-^y - D_0^y)\bar{z}_{1,ij}^N| = |b_{2,ij}(D_-^y(\mathcal{L}z_1)_{ij} - D_0^y(\mathcal{L}z_1)_{ij})| \leq \\ &\frac{C\varepsilon}{h_{y,j}} e^{-\frac{\beta_1(1-x_i)}{\varepsilon}} = \frac{C\varepsilon \max\{h_{x,i}, \varepsilon\}}{h_{y,j}} L^{(1)} S_{ij} \leq CL^{(1)} S_{ij}. \end{aligned}$$

This result combined with (i) and (ii) gives us

$$|L^{(1)}(L^{(1)} - L^{(0)})\bar{z}_{1,ij}^N| \leq CL^{(1)} S_{ij} \quad \text{on } \Omega^N. \quad (4.18)$$

The operator $L^{(1)}$ is monotone (see Lemma 4.2), therefore,

$$|(L^{(1)} - L^{(0)})\bar{z}_{1,ij}^N| \leq S_{ij} \quad \text{on } \Omega^N.$$

■

Lemma 4.10 will be used in the proof of the following lemma.

Lemma 4.11 *There is a constant C , independent of the meshsizes and ε , such that*

$$|z_{1,ij}^N| \leq CS_{ij} \quad \text{on } \bar{\Omega}^N.$$

Proof Using (4.3), (2.11) and Lemma 4.10 we obtain

$$\begin{aligned} |L^{(1)}z_{1,ij}^N| &= |2(\mathcal{L}z_1)_{ij} - L^{(0)}\bar{z}_{1,ij}^N| \leq |(\mathcal{L}z_1)_{ij}| + |(L^{(1)} - L^{(0)})\bar{z}_{1,ij}^N| \leq \\ &C_1\epsilon e^{-\beta_1(1-x_i)/\epsilon} + C_2S_{ij} \leq CL^{(1)}S_{ij} \quad \text{on } \Omega^N. \\ |z_{1,ij}^N| &= |z_{1,ij}| \leq Ce^{-\beta_1(1-x_i)/\epsilon} \leq CS_{ij} \quad \text{on } \Gamma^N. \end{aligned}$$

Choosing the barrier function

$$w_{ij} = C_w S_{ij}$$

and using Lemma 4.1 we have

$$|z_{1,ij}^N| \leq |w_{ij}| = C_w S_{ij} \quad \text{on } \bar{\Omega}^N.$$

■

Now, we are in a position to estimate $|z_1 - z_1^N|$.

(1) Let $0 \leq i \leq N/2, 0 \leq j \leq N$, then (2.8), Lemma 4.11 and Lemma 4.9 yield

$$|(z_1 - z_1^N)_{ij}| \leq |z_{1,ij}| + |z_{1,ij}^N| \leq C_1 e^{-\beta_1(1-x_i)/\epsilon} + C_w S_{ij} \leq CN^{-2}. \quad (4.19)$$

(2) Let $N/2 < i \leq N, 0 \leq j \leq N$, then $h_{x,i} = h_x$ and

$$\frac{h_x}{\epsilon} \leq CN^{-1} \ln N, \quad \sinh\left(\frac{\beta_1 h_x}{\epsilon}\right) \leq C \frac{\beta_1 h_x}{\epsilon} \leq CN^{-1} \ln N.$$

Using Corollary 4.2 and Lemma 4.6(b) we obtain

$$\begin{aligned} |(L^{(1)} - L^{(0)})(z_1 - \bar{z}_1^N)_{ij}| &\leq \frac{C_x}{\epsilon} (N^{-1} \ln N)^2 S_{ij} + C_y N^{-2} \ln N, \\ |(L^{(0)} - \mathcal{L})z_{1,ij}| &\leq \frac{C_x}{\epsilon} (N^{-1} \ln N)^2 S_{ij} + C_y N^{-2} \end{aligned}$$

and hence

$$|L^{(1)}(z_1 - z_1^N)_{ij}| \leq C(N^{-1} \ln N)^2 L^{(1)} S_{ij}, \quad \text{for } N/2 < i < N, 0 \leq j \leq N. \quad (4.20)$$

Using (4.19) and (4.4) we have

$$|(z_1 - z_1^N)_{ij}| \leq \begin{cases} 0 & i = N, 0 \leq j \leq N, \\ N^{-2} & i = N/2, 0 \leq j \leq N. \end{cases} \quad (4.21)$$

Now, according Lemma 4.1, (4.20) and (4.21) give us for $N/2 < i \leq N, 0 \leq j \leq N$,

$$|(z_1 - z_1^N)_{ij}| \leq C(N^{-1} \ln N)^2. \quad (4.22)$$

Combining (4.19) with (4.22) follows

$$|(z_1 - z_1^N)_{ij}| \leq C(N^{-1} \ln N)^2 \quad \text{on } \bar{\Omega}^N. \quad (4.23)$$

Analogously, we have the same result for the other boundary layer term

$$|(z_2 - z_2^N)_{ij}| \leq C(N^{-1} \ln N)^2 \quad \text{on } \bar{\Omega}^N. \quad (4.24)$$

(3) Corner layer part

The discrete equivalent of the boundary value term $e^{-\beta_1(1-x)/\varepsilon} e^{-\beta_2(1-y)/\varepsilon}$ is given by the function

$$\begin{cases} \bar{S}_{ij} = \prod_{k=i+1}^N (1 + \frac{\beta_1 h_{x,k}}{\varepsilon})^{-1} \prod_{k=j+1}^N (1 + \frac{\beta_2 h_{y,k}}{\varepsilon})^{-1} & \text{on } \Omega^N, \\ \bar{S}_{ij} = 1 & \text{on } \Gamma^N. \end{cases}$$

Definition 4.1

$$\begin{aligned} \Omega_0^N &= (0, 1 - \tau_x] \times (0, 1 - \tau_y] & \Omega_1^N &= (1 - \tau_x, 1) \times (0, 1 - \tau_y] \\ \Omega_2^N &= (0, 1 - \tau_x] \times (1 - \tau_y, 1) & \Omega_{12}^N &= (1 - \tau_x, 1) \times (1 - \tau_y, 1) \end{aligned}$$

The next lemma is similar to Lemma 4.9.

Lemma 4.12 *There is a constant C , independent of the meshsizes and ε , such that the grid function \bar{S}_{ij} satisfies the following inequalities:*

$$L^{(1)} \bar{S}_{ij} \geq C \bar{S}_{ij} \left(\frac{1}{\max\{h_{x,i}, \varepsilon\}} + \frac{1}{\max\{h_{y,j}, \varepsilon\}} \right) \quad \text{on } \Omega^N, \quad (4.25)$$

$$e^{-\frac{\beta_1(1-x_i)}{\varepsilon}} e^{-\frac{\beta_2(1-y_j)}{\varepsilon}} \leq \bar{S}_{ij} \quad \text{on } \Omega^N, \quad (4.26)$$

$$\bar{S}_{ij} \leq CN^{-2} \quad \text{on } \Omega^N \setminus \Omega_{12}^N. \quad (4.27)$$

Proof The proofs of (4.25), (4.26) and (4.27) directly follow from the corresponding inequalities in Lemma 4.9. \blacksquare

Corollary 4.3 *There are constants C_i , $i = 1, 2, 3, 4$, independent of the meshsizes and ε , such that*

$$|(L^{(1)} - L^{(0)})(z_{12} - \bar{z}_{12}^N)_{ij}| \leq \begin{cases} \frac{C_1}{\varepsilon^3} (H_x^2 + H_y^2) \bar{S}_{\frac{N}{2}, \frac{N}{2}}, & \text{on } \Omega_0^N, \\ \frac{C_2}{\varepsilon^2} (h_x \sinh(\frac{\beta_1 h_x}{\varepsilon}) + \frac{H_y^2}{\varepsilon}) \bar{S}_{i, \frac{N}{2}}, & \text{on } \Omega_1^N, \\ \frac{C_3}{\varepsilon^2} (\frac{H_x^2}{\varepsilon} + h_y \sinh(\frac{\beta_2 h_y}{\varepsilon})) \bar{S}_{\frac{N}{2}, j}, & \text{on } \Omega_2^N, \\ \frac{C_4}{\varepsilon^2} (h_x \sinh(\frac{\beta_1 h_x}{\varepsilon}) + h_y \sinh(\frac{\beta_2 h_y}{\varepsilon})) \bar{S}_{ij}, & \text{on } \Omega_{12}^N. \end{cases}$$

Proof Corollary 4.1(d), Lemma 4.12 and an analogous calculations of those in Corollary 4.2 yield the estimates in this lemma. \blacksquare

Lemma 4.13 *There is a constant C , independent of the meshsizes and ε , such that*

$$|(L^{(1)} - L^{(0)})\bar{z}_{12,ij}^N| \leq C\bar{S}_{ij} \quad \text{on } \Omega^N.$$

Proof Repeating the derivations from the proof of Lemma 4.10 (i)&(ii) we obtain

$$|L^{(1)}(L_x^{(1)} - L_x^{(0)})\bar{z}_{12,ij}^N| \leq \frac{C}{\max\{h_{x,i}, \varepsilon\}}\bar{S}_{ij} \quad \text{on } \Omega^N,$$

$$|L^{(1)}(L_y^{(1)} - L_y^{(0)})\bar{z}_{12,ij}^N| \leq \frac{C}{\max\{h_{y,j}, \varepsilon\}}\bar{S}_{ij} \quad \text{on } \Omega^N.$$

Using the triangular inequality and these results we have

$$|L^{(1)}(L^{(1)} - L^{(0)})\bar{z}_{12,ij}^N| \leq CL^{(1)}\bar{S}_{ij} \quad \text{on } \Omega^N. \quad (4.28)$$

The operator $L^{(1)}$ is monotone, therefore,

$$|(L^{(1)} - L^{(0)})\bar{z}_{12,ij}^N| \leq \bar{S}_{ij} \quad \text{on } \Omega^N.$$

Lemma 4.13 will be used in the proof of the following lemma. \blacksquare

Lemma 4.14 *There is a constant C , independent of the meshsizes and ε , such that*

$$|z_{12,ij}^N| \leq C\bar{S}_{ij} \quad \text{on } \bar{\Omega}^N.$$

Proof It is analogous to the proof of Lemma 4.11 \blacksquare

Now, we are in a position to estimate $|z_{12} - z_{12}^N|$.

- (1) Let us consider $|(z_{12} - z_{12}^N)_{ij}|$ on $\Omega^N \setminus \Omega_{12}^N$. Then, (2.10), Lemma 4.14 and Lemma 4.12 yield

$$|(z_{12} - z_{12}^N)_{ij}| \leq |z_{12,ij}| + |z_{12,ij}^N| \leq C_1 e^{-\beta_1(1-x_i)/\varepsilon} e^{-\beta_2(1-y_j)/\varepsilon} + C_w \bar{S}_{ij} \leq CN^{-2}. \quad (4.29)$$

- (2) Let us consider $|(z_{12} - z_{12}^N)_{ij}|$ on Ω_{12}^N . Then, $h_{x,i} = h_x$, $h_{y,j} = h_y$ and

$$\begin{aligned} \frac{h_x}{\varepsilon} &\leq CN^{-1} \ln N, \quad \sinh\left(\frac{\beta_1 h_x}{\varepsilon}\right) \leq C \frac{\beta_1 h_x}{\varepsilon} \leq CN^{-1} \ln N, \\ \frac{h_y}{\varepsilon} &\leq CN^{-1} \ln N, \quad \sinh\left(\frac{\beta_2 h_y}{\varepsilon}\right) \leq C \frac{\beta_2 h_y}{\varepsilon} \leq CN^{-1} \ln N. \end{aligned}$$

Using Corollary 4.3 and Lemma 4.6(d) we obtain

$$|L^{(1)}(z_{12} - z_{12}^N)_{ij}| \leq C(N^{-1} \ln N)^2 L^{(1)}\bar{S}_{ij} \quad \text{on } \Omega_{12}^N. \quad (4.30)$$

Using (4.29) and (4.4) we have

$$|(z_{12} - z_{12}^N)_{ij}| \leq \begin{cases} 0 & \text{on } \Gamma^N, \\ N^{-2} & \text{for } i = N/2 \text{ or } j = N/2. \end{cases} \quad (4.31)$$

Now, according Lemma 4.1, (4.30) and (4.31) give us

$$|(z_{12} - z_{12}^N)_{ij}| \leq C(N^{-1} \ln N)^2 \quad \text{on } \Omega_{12}^N. \quad (4.32)$$

Combining (4.29) with (4.32) and $z_{12} = z_{12}^N$ on Γ implies

$$|(z_{12} - z_{12}^N)_{ij}| \leq C(N^{-1} \ln N)^2 \quad \text{on } \bar{\Omega}^N. \quad (4.33)$$

We summarize (4.14), (4.23), (4.24), (4.33) in the following theorem.

Theorem 4.1 *Let u be the solution of equation (1.1)-(1.2) and u^N be its discrete solution obtained by applying the second order defect-correction method on the Shishkin mesh Ω^N , where the number of the points N at each direction satisfies (4.13). Let the conditions of Theorem 2.1 hold true and the velocity vector \mathbf{b} satisfy (1.3). Then, there is a constant C , independent of the meshsizes and ε , such that the discretization error is globally uniformly bounded in ε by*

$$\|u - u^N\| \leq C(N^{-1} \ln N)^2,$$

where the total number of points used is N^2 .

Remark 4.1 Observe that the discretization error as well as the total number of points used do not depend on ε . The influence of ε appears only in the choice of the transition points and the minimal meshsizes used to resolve the layers.

5 Numerical results

In this section we shall consider three numerical examples in order to verify the theory in Section 4 and to illustrate how efficiently the Shishkin mesh resolves the exponential boundary layers.

Since the problems are nonsymmetric and ill-conditioned, we use GCG-MR (Generalized Conjugate Gradient Minimal Residual) iterative solver preconditioned by block (M)ILU factorization, as presented in [3].

The following problems will be considered,

$$\begin{aligned} -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu &= f, & (x, y) \in \Omega = (0, 1)^2, \\ u &= 0, & (x, y) \in \Gamma = \partial\Omega, \end{aligned}$$

where

$$\mathbf{1) } \quad \varepsilon \in [10^{-9}, 10^{-3}], \quad c = 1, \quad \mathbf{b} = [2 + (1 - x)^2, 1 + xy] \quad \text{and} \quad f(x, y) = 8xy^2;$$

- 2) $\varepsilon \in [10^{-8}, 10^{-3}]$, $c = 1$, $\mathbf{b} = [2 + x^2y, 1 + xy]$ and $f(x, y) = 4x + y^2$;
- 3) $\varepsilon \in [10^{-8}, 10^{-3}]$, $c = 1$, $\mathbf{b} = [\frac{\sqrt{2}}{2}(1 - \frac{\sqrt{2}}{2}x), \frac{\sqrt{2}}{2}(1 - \frac{\sqrt{2}}{2}y)]$ and $f(x, y)$ is such that the exact solution is:

$$u(x, y) = x^2y^2(1 - e^{-\frac{1-x}{\varepsilon}})(1 - e^{-\frac{1-y}{\varepsilon}}). \quad (5.1)$$

The exact solutions of Problems 1 and 2 are not known that is why we use two successive meshes Ω^N and Ω^{2N} to estimate the discretization error

$$\|e^N\| = \max_{i,j} |u_{ij}^N - u_{2i2j}^{2N}|, \quad i, j = 0, 1, \dots, N,$$

where u^{2N} is the approximate solution on the mesh $\Omega^{2N} = \{(x_i, y_j), i, j = 0, 1, \dots, 2N\}$ defined by

$$\begin{cases} (x_{2i}, y_{2j}) = (x_i, y_j) \in \Omega^N, & i, j = 0, 1, \dots, N, \\ (x_{2i+1}, y_{2j+1}) = (\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}), & i, j = 0, 1, \dots, N-1. \end{cases}$$

Four tables are given for each example, the first two display the L_∞ -norm of the global discretization errors of the upwind and the defect-correction schemes for $N = 2^4, 2^5, 2^6, 2^7$ and different ε , while the following two tables display the L_∞ -norm of the same discretization errors but only on the coarse mesh Ω_0^N , where the solution is assumed to be smooth. All numerical results clearly illustrate the uniform convergence rate in the perturbation parameter ε . The essential reduction of the upwind discretization error due to the corrections δ^N is clearly seen in the tables of the defect-correction discretization error.

The last three columns of each table show the rate of convergence

$$rate_conv = (\ln\|e^N\| - \ln\|e^{2N}\|)/\ln 2$$

for $N = 2^4, 2^5, 2^6$. The value expected for $rate_conv$, i.e. exp_rate_conv , is given in the brackets at the top of each column. It is 1 and 2, correspondingly for the upwind scheme and the defect-correction method in the smooth part of the solution and it is calculated in the layer part using the formula

$$exp_rate_conv = \left(\ln \left(\frac{N^{-1} \ln N}{(2N)^{-1} \ln(2N)} \right)^s \right) / \ln 2 = \left(\ln \left(\frac{2k}{k+1} \right)^s \right) / \ln 2,$$

where $N = 2^k$, $k = 4, 5, 6$ and $s = \begin{cases} 1, & \text{for the upwind scheme,} \\ 2, & \text{for the defect-correction scheme.} \end{cases}$

Examples 1 and 2 are taken from [15] in order to compare three numerical methods - **the upwind scheme**, presented in [15] and here as a first step of the defect-correction method, **the hybrid scheme**, presented in [15], and **the defect-correction scheme**, presented in this paper. In the hybrid scheme the first order derivatives of (1.1) are approximated by upwind differences in the smooth part of the solution and with central differences in the layer part of the solution, where the local meshsizes are in

Upwind scheme, Shishkin mesh							
ε	global discretization error on Ω^N				rate of convergence		
	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (0.678)	$N = 32$ (0.737)	$N = 64$ (0.778)
10^{-3}	6.926875e-02	5.358409e-02	3.776375e-02	2.477237e-02	0.370	0.505	0.608
10^{-4}	6.929696e-02	5.361500e-02	3.779218e-02	2.479615e-02	0.370	0.505	0.608
10^{-5}	6.929942e-02	5.361777e-02	3.779478e-02	2.479847e-02	0.370	0.505	0.608
10^{-6}	6.929966e-02	5.361805e-02	3.779504e-02	2.479874e-02	0.370	0.505	0.608
10^{-7}	6.929969e-02	5.361806e-02	3.779502e-02	2.479896e-02	0.370	0.505	0.608
10^{-8}	6.929975e-02	5.361857e-02	3.779598e-02	2.479916e-02	0.370	0.504	0.608
10^{-9}	6.929988e-02	5.361813e-02	3.779171e-02	2.479976e-02	0.370	0.505	0.608

Table 1: Example 1

general much smaller than ε .

Based on the errors obtained by these three schemes for fixed N and ε , they can be graded in the following way - the upwind scheme, the hybrid scheme which gives better results and the defect-correction scheme, which gives the best results.

According Theorem 3.2 in [9] the solutions of Example 1 and 2 only belong to $\mathbb{C}^{1,1}(\bar{\Omega})$, while the theoretical analysis presented in this paper requires $u \in \mathbb{C}^4(\bar{\Omega})$. Despite of this fact we got for Example 1 convergence rates in the smooth and layer parts of the solution close to the expected one. All experiments show that the maximal global error is in the layer part, see Figures 1(b), 2(b) and 3(a).

The approximate solutions of Examples 1,2 are plotted in Figures 1(a), 2(a), correspondingly. The solution u of Example 2 behaves worse on the coarse mesh Ω_0^N than the solution of Example 1 since the second order compatibility conditions (2.2) are not satisfied at the origin and consequently a singularity in the derivatives of u pollutes the subdomain where the solution is assumed to be smooth and a coarse mesh Ω_0^N is used. In practise, we do not have a pure smooth part as in Example 1 since the singularity propagates across Ω_0^N . This is clearly illustrated in Figure 2(b) and Tables 7, 8, where the *rate_conv* is much less than the *exp_rate_conv*.

The righthand side of Example 3 lies in $\mathbb{C}(\bar{\Omega})$ and satisfies the first order compatibility conditions (2.1). Thus, $u \in \mathbb{C}^3(\bar{\Omega})$ and due to this smoothness we got in both parts $\Omega^N \setminus \Omega_{12}^N$ and Ω_{12}^N , where a coarse and a fine mesh is used correspondingly, a rate of convergence close to the predicted one by the theory in Section 4. The maximal error is obtained next to the outflow boundary, which is illustrated in Figure 3(b), where the discretization error on Ω_{12}^N is shown.

6 Conclusions

A finite difference method for singularly perturbed convection-diffusion problems has been presented, based on a defect-correction method on a Shishkin mesh, which is a priori adapted mesh to the behavior of the solution with exponential layers. This method is chosen in order to avoid the disadvantages of the classical central and

	Defect-Correction method, Shishkin mesh						
	global discretization error on Ω^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (1.356)	$N = 32$ (1.474)	$N = 64$ (1.555)
10^{-3}	4.122014e-02	1.602466e-02	6.000329e-03	2.170978e-03	1.363	1.417	1.467
10^{-4}	4.179315e-02	1.623213e-02	6.073407e-03	2.195806e-03	1.364	1.418	1.468
10^{-5}	4.185073e-02	1.625328e-02	6.080728e-03	2.198288e-03	1.365	1.418	1.468
10^{-6}	4.185649e-02	1.625541e-02	6.081462e-03	2.198524e-03	1.365	1.418	1.468
10^{-7}	4.185707e-02	1.625563e-02	6.081525e-03	2.198545e-03	1.365	1.418	1.468
10^{-8}	4.185713e-02	1.625564e-02	6.081566e-03	2.198612e-03	1.365	1.418	1.468
10^{-9}	4.185718e-02	1.625584e-02	6.082941e-03	2.198678e-03	1.365	1.418	1.468

Table 2: Example 1

	Upwind scheme, Shishkin mesh						
	discretization error on the coarse mesh Ω_0^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (1.000)	$N = 32$ (1.000)	$N = 64$ (1.000)
10^{-3}	2.131695e-02	1.088535e-02	5.718715e-03	2.994195e-03	0.970	0.929	0.934
10^{-4}	2.162705e-02	1.106788e-02	5.857582e-03	3.079034e-03	0.966	0.918	0.928
10^{-5}	2.165829e-02	1.108633e-02	5.871586e-03	3.087592e-03	0.966	0.917	0.927
10^{-6}	2.166142e-02	1.108817e-02	5.872987e-03	3.088448e-03	0.966	0.917	0.927
10^{-7}	2.166173e-02	1.108836e-02	5.873129e-03	3.088476e-03	0.966	0.917	0.927
10^{-8}	2.166176e-02	1.108832e-02	5.873022e-03	3.088494e-03	0.966	0.917	0.927
10^{-9}	2.166136e-02	1.108927e-02	5.872858e-03	3.088612e-03	0.966	0.917	0.927

Table 3: Example 1

	Defect-Correction method, Shishkin mesh						
	discretization error on the coarse mesh Ω_0^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (2.000)	$N = 32$ (2.000)	$N = 64$ (2.000)
10^{-3}	6.432830e-03	1.496443e-03	4.433505e-04	1.200299e-04	2.104	1.755	1.885
10^{-4}	6.578734e-03	1.554725e-03	4.475479e-04	1.218630e-04	2.081	1.797	1.877
10^{-5}	6.593597e-03	1.560836e-03	4.478882e-04	1.216608e-04	2.079	1.801	1.880
10^{-6}	6.595087e-03	1.561449e-03	4.479218e-04	1.216704e-04	2.079	1.802	1.880
10^{-7}	6.595237e-03	1.561515e-03	4.479283e-04	1.216776e-04	2.078	1.802	1.880
10^{-8}	6.595264e-03	1.561497e-03	4.480567e-04	1.216819e-04	2.079	1.801	1.881
10^{-9}	6.595298e-03	1.561671e-03	4.499133e-04	1.216912e-04	2.078	1.795	1.886

Table 4: Example 1

	Upwind scheme, Shishkin mesh						
	global discretization error on Ω^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (0.678)	$N = 32$ (0.737)	$N = 64$ (0.778)
10^{-3}	7.586772e-02	5.543866e-02	3.810361e-02	2.481169e-02	0.453	0.541	0.619
10^{-4}	7.596598e-02	5.545016e-02	3.811281e-02	2.482380e-02	0.454	0.541	0.619
10^{-5}	7.597576e-02	5.545124e-02	3.811372e-02	2.482500e-02	0.454	0.541	0.619
10^{-6}	7.597674e-02	5.545135e-02	3.811381e-02	2.482515e-02	0.454	0.541	0.619
10^{-7}	7.597684e-02	5.545136e-02	3.811388e-02	2.482539e-02	0.454	0.541	0.618
10^{-8}	7.597665e-02	5.545124e-02	3.811243e-02	2.482585e-02	0.454	0.541	0.618

Table 5: Example 2

	Defect-Correction method, Shishkin mesh						
	global discretization error on Ω^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (1.356)	$N = 32$ (1.474)	$N = 64$ (1.555)
10^{-3}	4.997354e-02	2.070080e-02	7.390237e-03	2.511611e-03	1.271	1.486	1.557
10^{-4}	5.050584e-02	2.089910e-02	7.419671e-03	2.520730e-03	1.273	1.494	1.558
10^{-5}	5.055932e-02	2.091891e-02	7.422493e-03	2.521687e-03	1.273	1.495	1.558
10^{-6}	5.056467e-02	2.092089e-02	7.422777e-03	2.521785e-03	1.273	1.495	1.558
10^{-7}	5.056520e-02	2.092108e-02	7.422793e-03	2.521731e-03	1.273	1.495	1.558
10^{-8}	5.056531e-02	2.092111e-02	7.422731e-03	2.521751e-03	1.273	1.495	1.558

Table 6: Example 2

	Upwind scheme, Shishkin mesh						
	discretization error on the coarse mesh Ω_0^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (1.000)	$N = 32$ (1.000)	$N = 64$ (1.000)
10^{-3}	3.371877e-02	2.348215e-02	1.411174e-02	7.811470e-03	0.522	0.735	0.853
10^{-4}	3.424953e-02	2.398239e-02	1.447805e-02	8.039902e-03	0.514	0.728	0.849
10^{-5}	3.430298e-02	2.403294e-02	1.451515e-02	8.062787e-03	0.513	0.727	0.848
10^{-6}	3.430833e-02	2.403800e-02	1.451886e-02	8.065075e-03	0.513	0.727	0.848
10^{-7}	3.430886e-02	2.403851e-02	1.451923e-02	8.065309e-03	0.513	0.727	0.848
10^{-8}	3.430891e-02	2.403856e-02	1.451925e-02	8.065459e-03	0.513	0.727	0.848

Table 7: Example 2

	Defect-Correction method, Shishkin mesh						
	discretization error on the coarse mesh Ω_0^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (2.000)	$N = 32$ (2.000)	$N = 64$ (2.000)
10^{-3}	2.186754e-02	1.094737e-02	4.694392e-03	1.939862e-03	0.998	1.222	1.275
10^{-4}	2.239554e-02	1.138135e-02	4.993345e-03	2.163513e-03	0.977	1.189	1.207
10^{-5}	2.244900e-02	1.142592e-02	5.024790e-03	2.188234e-03	0.974	1.185	1.199
10^{-6}	2.245435e-02	1.143039e-02	5.027951e-03	2.190732e-03	0.974	1.185	1.199
10^{-7}	2.245488e-02	1.143084e-02	5.028265e-03	2.190979e-03	0.974	1.185	1.198
10^{-8}	2.245494e-02	1.143089e-02	5.028265e-03	2.191011e-03	0.974	1.185	1.198

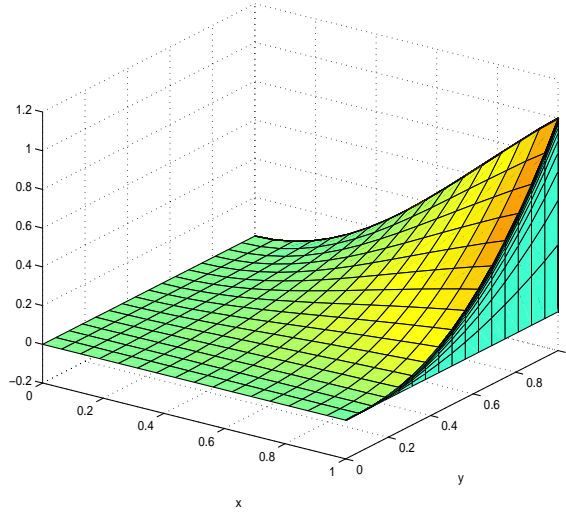
Table 8: Example 2

	Upwind scheme, Shishkin mesh						
	global discretization error on Ω^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (0.678)	$N = 32$ (0.737)	$N = 64$ (0.778)
10^{-3}	4.799874e-01	2.778214e-01	1.533847e-01	8.076659e-02	0.789	0.857	0.925
10^{-4}	4.919392e-01	2.838387e-01	1.565635e-01	8.237354e-02	0.793	0.858	0.926
10^{-5}	4.931915e-01	2.844680e-01	1.569083e-01	8.253771e-02	0.794	0.858	0.927
10^{-6}	4.933171e-01	2.845312e-01	1.569429e-01	8.255416e-02	0.794	0.858	0.927
10^{-7}	4.933279e-01	2.845375e-01	1.569464e-01	8.255580e-02	0.794	0.858	0.927
10^{-8}	4.933111e-01	2.845382e-01	1.569467e-01	8.255594e-02	0.794	0.858	0.927

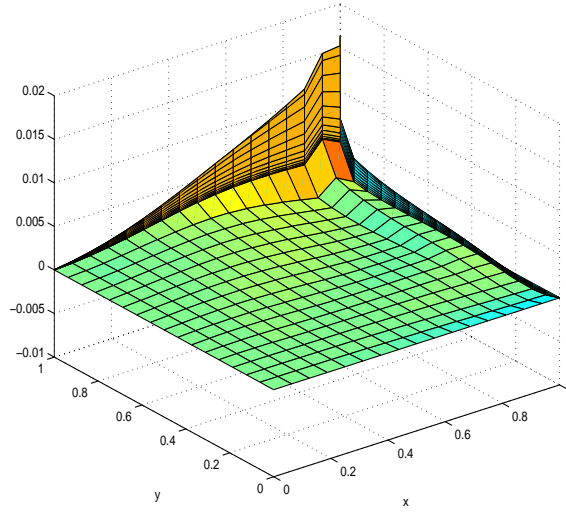
Table 9: Example 3

	Defect-Correction method, Shishkin mesh						
	global discretization error on Ω^N				rate of convergence		
ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (1.356)	$N = 32$ (1.474)	$N = 64$ (1.555)
10^{-3}	3.446238e-01	1.669894e-01	7.041923e-02	2.513839e-02	1.045	1.246	1.486
10^{-4}	3.544970e-01	1.710080e-01	7.177080e-02	2.572304e-02	1.052	1.253	1.480
10^{-5}	3.555245e-01	1.714234e-01	7.190904e-02	2.578279e-02	1.052	1.253	1.480
10^{-6}	3.556274e-01	1.714651e-01	7.192289e-02	2.578877e-02	1.052	1.253	1.480
10^{-7}	3.556355e-01	1.714692e-01	7.192427e-02	2.578937e-02	1.052	1.253	1.480
10^{-8}	3.556138e-01	1.714696e-01	7.192441e-02	2.578943e-02	1.052	1.253	1.480

Table 10: Example 3

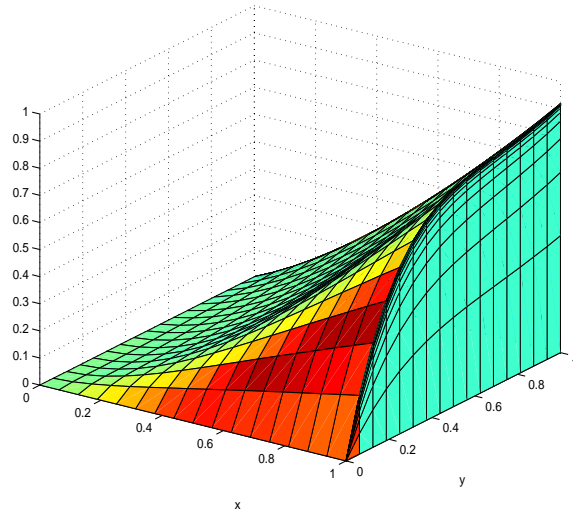


(a) Approximate solution

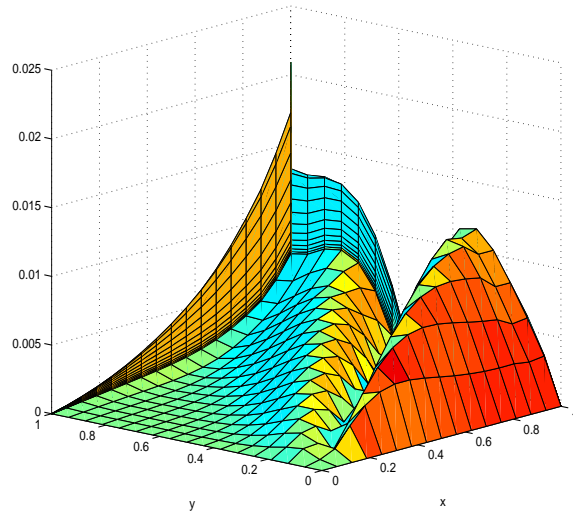


(b) Discr. error, $\|e^N\|_\infty = 1.625541e - 02$

Figure 1: Example 1, $\varepsilon = 10^{-6}$, D-C method on Shishkin mesh, $N = 32$.

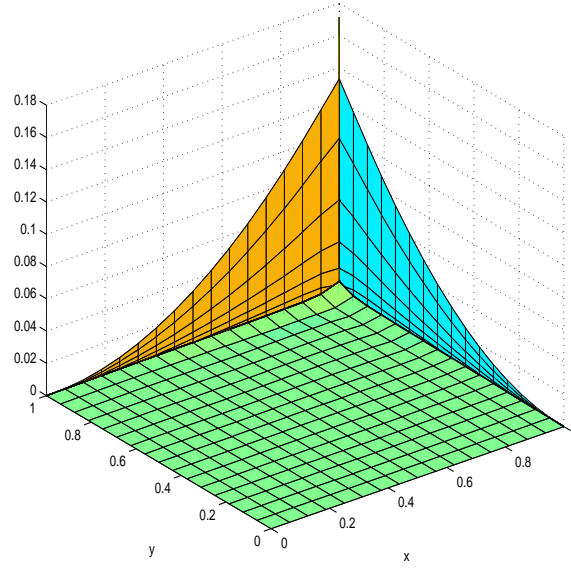


(a) Approximate solution

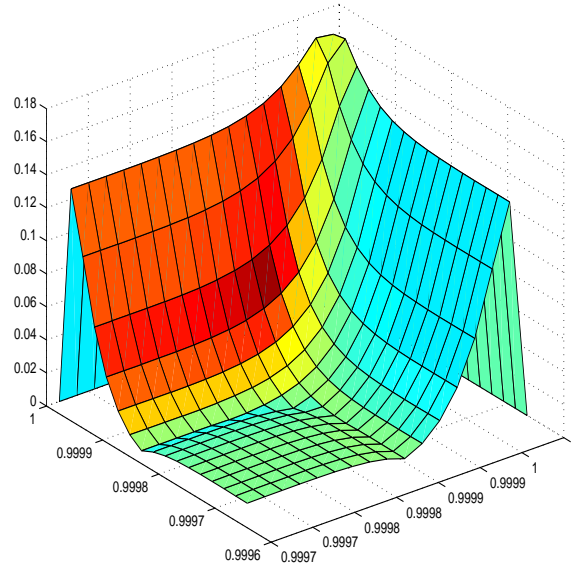


(b) Discr. error, $\|e^N\|_\infty = 2.092089e - 02$

Figure 2: Example 2, $\varepsilon = 10^{-6}$, D-C method on Shishkin mesh, $N = 32$.



(a) Discr. error, $\|e^N\|_\infty = 1.714234e - 01$



(b) Zoom of the layer part in Fig. 3(a)

Figure 3: Example 3, $\varepsilon = 10^{-5}$, D-C method on Shishkin mesh, $N = 32$.

Upwind scheme, Shishkin mesh							
ε	discretization error on the coarse mesh Ω_0^N				rate of convergence		
	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (1.000)	$N = 32$ (1.000)	$N = 64$ (1.000)
10^{-3}	3.192691e-02	1.739480e-02	8.882709e-03	4.420405e-03	0.876	0.970	1.007
10^{-4}	3.438306e-02	1.895122e-02	9.797620e-03	4.953349e-03	0.859	0.952	0.984
10^{-5}	3.463703e-02	1.911108e-02	9.888458e-03	5.004265e-03	0.858	0.951	0.983
10^{-6}	3.466252e-02	1.912712e-02	9.897528e-03	5.009304e-03	0.858	0.950	0.982
10^{-7}	3.466507e-02	1.912872e-02	9.898435e-03	5.009807e-03	0.858	0.950	0.982
10^{-8}	3.466533e-02	1.912888e-02	9.898525e-03	5.009857e-03	0.858	0.950	0.982

Table 11: Example 3

Defect-Correction method, Shishkin mesh							
ε	discretization error on the coarse mesh Ω_0^N				rate of convergence		
	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 16$ (2.000)	$N = 32$ (2.000)	$N = 64$ (2.000)
10^{-3}	1.682246e-03	5.778592e-04	1.862772e-04	5.473805e-05	1.542	1.633	1.767
10^{-4}	2.392108e-03	6.642367e-04	2.164592e-04	6.511334e-05	1.849	1.618	1.733
10^{-5}	2.481380e-03	6.778238e-04	2.213429e-04	6.627140e-05	1.872	1.615	1.740
10^{-6}	2.490323e-03	6.792157e-04	2.219571e-04	6.648547e-05	1.874	1.614	1.739
10^{-7}	2.489182e-03	6.793553e-04	2.220184e-04	6.650630e-05	1.873	1.613	1.739
10^{-8}	2.468699e-03	6.793692e-04	2.220246e-04	6.650838e-05	1.861	1.613	1.739

Table 12: Example 3

upwind difference operators. It combines the stability property of the lower order operator (upwind) with the high order of accuracy of the unstable operator (central). This gives a difference method of order $O(N^{-2} \ln^2 N)$, uniformly in ε , where the total number of points used is N^2 . The proof of the latter convergence result, which has not been proved earlier in the literature, is given in Section 4. The numerical results in Section 5 indicate nearly second order of convergence in agreement with the theoretical result. The defect-correction technique makes easy the solution of the arising linear systems of equations since only a M matrix is involved. Moreover, as it has been shown in [2], it provides excellent error indicators for a construction of a posteriori adapted meshes.

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